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ROTATIONAL LINEAR WEINGARTEN SURFACES OF HYPERBOLIC TYPE

BY

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ABSTRACT

A linear Weingarten surface in Euclidean space $\mathbb{R}^3$ is a surface whose mean curvature H and Gaussian curvature K satisfy a relation of the form $aH + bK = c$, where $a, b, c \in \mathbb{R}$. Such a surface is said to be hyperbolic when $a^2 + 4bc < 0$. In this paper we study rotational linear Weingarten surfaces of hyperbolic type giving a classification under suitable hypothesis. As a consequence, we obtain a family of complete hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$ that consists of surfaces with self-intersections whose generating curves are periodic.

1. Introduction

A surface S in a 3-dimensional Euclidean space $\mathbb{R}^3$ is called a **Weingarten surface** if there is some relation between its two principal curvatures κ_1 and κ_2, that is, if there is a smooth function W of two variables such that $W(\kappa_1, \kappa_2) = 0$. In particular, if K and H denote the Gauss and the mean curvature of S, respectively; the identity $W(\kappa_1, \kappa_2) = 0$ implies a relation $U(K, H) = 0$. Weingarten introduced this kind of surfaces in the context of the problem of finding all surfaces isometric to a given surface of revolution [14], [15]. In this paper we study Weingarten surfaces that satisfy the simplest case for U, that is, that U is of the linear type

$$a\ H + b\ K = c,$$

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where $a, b, c \in \mathbb{R}$. We say that S is a **linear Weingarten surface** and we abbreviate by LW-surface. First examples of LW-surfaces are the surfaces with constant mean curvature ($b = 0$) and the surfaces with constant Gauss curvature ($a = 0$). Although these two kinds of surfaces have been extensively studied in the literature, the classification of LW-surfaces in the general case is almost completely open today. Along the history, they have been of interest for geometers, mainly when the surface is closed: [2], [3], [8], [9], [10], [11], [13].

The behavior of a LW-surface and its qualitative properties strongly depend on the sign of the discriminant $\Delta := a^2 + 4bc$. Such a surface is said to be hyperbolic (resp., elliptic) when $\Delta < 0$ (resp., $\Delta > 0$). The relation $\Delta = 0$ characterizes the tubular surfaces. Examples of elliptic surfaces are the surfaces with constant mean curvature and the surfaces with positive constant Gaussian curvature. Since elliptic LW-surfaces have similar properties as these two kinds of surfaces, they have been of interest for a number of authors. For example, elliptic LW-surfaces satisfy a maximum principle and this enables the use of the Alexandrov reflection technique in its study. See the recent bibliography [1], [4], [7], [12].

The aim of this paper is the study of the LW-surfaces of hyperbolic type. Examples of hyperbolic LW-surfaces are the surfaces with negative constant Gaussian curvature ($a = 0, bc < 0$). One expects then to find in the hyperbolic LW-surfaces properties similar to those of the surfaces with negative constant Gaussian curvature. However, as we shall see, our family is more extensive and richer even in the rotational case; for example, we will obtain rotational hyperbolic LW-surfaces with positive Gaussian curvature (Theorem 3.2). Notice that in a hyperbolic Weingarten surface, the condition $\Delta < 0$ implies that umbilical points do not exist on the surface. As $a^2 + 4bc < 0$, it follows that $c \neq 0$. Without loss of generality, throughout this work we shall assume that $c = 1$ and the linear Weingarten relation is now

$$a\ H + b\ K = 1. \tag{1}$$

Among all hyperbolic LW-surfaces, the class of the surfaces of revolution are particularly interesting because in such case, equation (1) leads to an ordinary differential equation. Its study is then reduced to finding the profile curve that defines the surface. In this paper, we classify rotational hyperbolic LW-surfaces with suitable hypothesis on the profile curve. We summarize our classification as follows (see Theorems 3.2, 4.1, 5.2 and 6.4):

Let a and b be real numbers under the condition $a^2 + 4b < 0$. We consider the family of rotational linear Weingarten surfaces satisfying $aH + bK = 1$ whose profile curve presents a point whose tangent line is parallel to the axis of revolution. Then this family of surface can be parametrized by one parameter z_0, namely, $S(a, b; z_0)$, with $a, z_0 > 0$ and $z_0 \neq -2b/a$ such that: (i) if $0 < z_0 < a/2$, the surface is not complete and with positive Gaussian curvature; (ii) if $z_0 = a/2$, the surface is a right cylinder; (iii) if $a/2 < z_0 < -2b/a$, the surface is not complete and with negative Gaussian curvature; (iv) if $z_0 > -2b/a$, the surface is complete and periodic.

In Figure 1 we present a scheme of the classification of rotational hyperbolic LW-surfaces realized in this paper. In order to get this classification, we shall describe the symmetries and qualitative properties of these surfaces. Among the rotational hyperbolic LW-surfaces obtained in the above result, a special class of such surfaces is the family of surfaces $S(a, b; z_0)$ with $z_0 > -2b/a$, which have remarkable properties. For this reason, we separate and emphasize the statement as follows (Corollary 6.6):

There exists a one-parameter family of rotational hyperbolic linear Weingarten surfaces that are complete and with self-intersections in $\mathbb{R}^3$. Moreover, the profile curves are periodic.

This contrasts with Hilbert's theorem that there do not exist complete surfaces with constant negative Gaussian curvature immersed in $\mathbb{R}^3$. See also [4] for other examples.

This paper is organized as follows. Section 2 introduces notation and terminology used throughout this paper. Sections 3 to 6 successively describe the properties of rotational hyperbolic LW-surfaces according the value assigned to the parameter z_0. For each case depending on z_0, we present pictures of the profile curves. Finally, in Section 7, we study LW-surfaces whose profile curve has no tangent lines parallel to the axis of revolution and thus, they are not comprised in any of the previous sections.

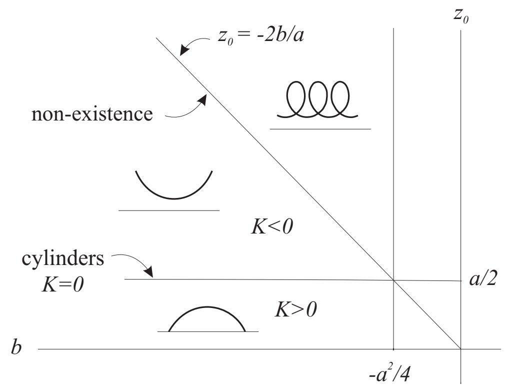


Figure 1. Classification of rotational hyperbolic LW-surfaces. We fix the value of the constant a in equation (1) with $a > 0$. The diagram represents the (b, z_0)-plane with $z_0 = z_0(b)$. If $z_0 = -2b/a$, do not exist LW-surfaces. When $z_0 \neq -2b/a$, we have four families of LW-surfaces depending if $z_0 = a/2$ or if z_0 belongs to the intervals $(0, a/2)$, $(a/2, -2b/a)$ and $(-2b/a, \infty)$.

2. Preliminaries

Let $\mathbb{R}^3$ be the three-dimensional space with usual coordinates (x, y, z). Let $\alpha : I \to \mathbb{R}^3$ be a planar curve in the (x, z)-plane with coordinate functions $\alpha(s) = (x(s), 0, z(s))$ and $z(s) > 0$. Assume that s is the arclength along α. Consider $\theta = \theta(s)$ the angle function that makes the velocity $\alpha'(s)$ at s with the x-axis, that is, $\alpha'(s) = (\cos\theta(s), 0, \sin\theta(s))$. The curvature of the planar curve α is given by θ'. Let S be the surface of revolution obtained by rotating α with respect to the x-axis, that is, S parametrizes as $X(s, \phi) = (x(s), z(s)\cos\phi, z(s)\sin\phi)$. The principal curvatures of S are given by

$$\kappa_1(s, \phi) = \cos\theta(s)/z(s), \quad \kappa_2(s, \phi) = -\theta'(s),$$

and the mean curvature and the Gaussian curvature of S write, respectively, as

$$H(s, \phi) = \frac{\cos\theta(s) - z(s)\theta'(s)}{2z(s)}, \quad K(s, \phi) = -\frac{\cos\theta(s)\theta'(s)}{z(s)}. \tag{2}$$

The Weingarten relation (1) converts into

$$a\,\frac{\cos\theta(s) - z(s)\theta'(s)}{2z(s)} - b\,\frac{\cos\theta(s)\theta'(s)}{z(s)} = 1. \tag{3}$$

Moreover, throughout this work, we disregard the fact that the surface has negative constant Gaussian curvature. Thus, we assume $a \neq 0$.

The study of rotational hyperbolic LW-surfaces reduces to the knowledge of the solutions of equation (3) for given initial data. Assume that at $s = 0$ the tangent line to α is parallel to the axis of revolution. Thus $\alpha(0) = (0, 0, z_0)$, $z_0 > 0$ and $\alpha'(0) = (1, 0, 0)$. Then the curve α is governed by the system of differential equations

$$\begin{cases} x'(s) = \cos\theta(s) \\ z'(s) = \sin\theta(s) \\ \theta'(s) = \frac{a\cos\theta(s) - 2z(s)}{az(s) + 2b\cos\theta(s)} \end{cases} \tag{4}$$

with initial conditions

$$x(0) = 0, \qquad z(0) = z_0, \qquad \theta(0) = 0. \tag{5}$$

If necessary, we shall denote by $\alpha(s; z_0)$ the solution obtained in (4)–(5) to emphasize the dependence on the parameter z_0. We have assumed that $z(s) > 0$ in the parametrization of α. However, if we replace the generating curve α by $\bar{\alpha}(s) = (x(s), 0, -z(s))$, the surface obtained by rotating the new curve $\bar{\alpha}$ with respect to the x-axis is the same that the α one, but now the curve $\bar{\alpha}$ is the solution of (4)–(5) changing the parameters (a, b, z_0) by $(-a, b, -z_0)$. Hence, we can choose the parameters a and z_0 to have the same sign. For convenience, we shall assume that both a and z_0 are positive numbers.

Theorem 2.1: *A first integral of the differential equations system* (4)–(5) *is given by*

$$z(s)^2 - az(s)\cos\theta(s) - b\cos^2\theta(s) - (z_0^2 - az_0 - b) = 0. \tag{6}$$

Proof. Multiplying both sides of (3) by $\sin\theta$, we obtain a first integral of type

$$z(s)^2 - az(s)\cos\theta(s) + b\sin^2\theta(s) + \lambda = 0$$

for some $\lambda \in \mathbb{R}$. At $s = 0$, we conclude $\lambda = az_0 - z_0^2$. ∎

We will also write (6) in the form

$$z(s)^2 - az(s)\cos\theta(s) + b\sin^2\theta(s) + az_0 - z_0^2 = 0. \tag{7}$$

We show that our solutions have symmetries at the critical points of the function z.

Theorem 2.2 (Symmetry): *Let $\alpha(s) = (x(s), 0, z(s))$ be the profile curve of a rotational hyperbolic LW-surface S where α is a solution of ((4)). If for some $s_1 \in \mathbb{R}$, $\sin\theta(s_1) = 0$, then α is symmetric with respect to the line $x = x(s_1)$.*

Proof. By hypothesis, $z'(s_1) = 0$. Without loss of generality, we assume that $x(s_1) = 0$. Then it suffices to show

$$\begin{aligned} x(s_1 - s) &= -x(s + s_1) \\ z(s_1 - s) &= z(s + s_1) \\ \theta(s_1 - s) &= -\theta(s + s_1) \end{aligned}$$

But each pair of the three functions is a solution of the same differential equations system for the same initial conditions. The uniqueness of solutions finishes the proof. ∎

We end this section by describing the phase portrait of the system of differential equations (4), which allows an understanding of the evolution of this system. Due to the periodicity of the cosine and sine functions, it suffices to study the system (4) for $\theta \in [0, 2\pi]$. We project the vector field (4) into the (θ, z)-plane, that is,

$$\begin{aligned} \theta'(s) &= \frac{a\cos\theta(s) - 2z(s)}{az(s) + 2b\cos\theta(s)} \\ z'(s) &= \sin\theta(s) \end{aligned}$$

In the region $[0, 2\pi] \times \{(\theta, z) : z > 0\}$, the vector field has exactly two singularities at the points $(0, a/2)$ and $(2\pi, a/2)$ and there is a curve where the vector field (θ', z') is not defined, namely,

$$\{(\theta, z = (-2b/a)\cos\theta) : 0 < \theta < \pi/2, 3\pi/2 < \theta < 2\pi\}.$$

Both singularities are saddle points because the two eigenvalues of the linearization at the singularities have opposite signs. Exactly, the two eigenvalues are $\pm 2/\sqrt{-\Delta}$, the singularities are of hyperbolic type and the eigenvectors of the linearized system are orthogonal. A sketch of the phase portrait of the system (4) appears in Figure 2.

Among the qualitative properties that we deduce from the phase portrait, we have the following ones depending on the range of the initial value z_0 in (5):

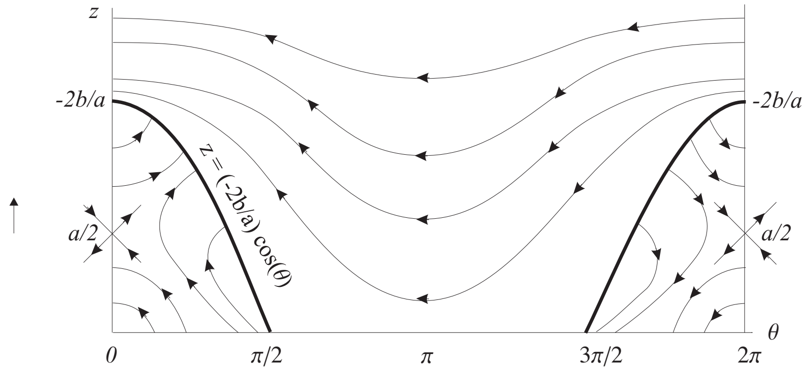


Figure 2. The phase portrait of the system of differential equations (4).

1. If $0 < z_0 < /2$, the generating curve α is defined in some bounded interval. Since the derivatives are bounded, the function z vanishes at the end points of the interval.
2. If $z_0 = a/2$, we have a stationary solution, that is, α is a straight-line.
3. If $a/2 < z_0 < -2b/a$, the solution exploits when z approaches $-2b/a\cos\theta$. This means that the maximal domain of the solution is some bounded interval.
4. If $z_0 > -2b/a$, it follows that the curve α is periodic.

According to this classification and depending if both numerator and denominator of θ' in (4) vanish, we distinguish between four cases depending on the initial value of z_0, namely, (i) $0 < z_0 < a/2$; (ii) $z_0 = a/2$; (iii) $a/2 < z_0 < -2b/a$ and (iv) $z_0 > -2b/a$. We point out that if $z_0 = -2b/a$, there are no solutions for (4)–(5).

3. The case $0 < z_0 < a/2$

In this section we discuss the first case of the value z_0 in (5): we assume,

$$0 < z_0 < a/2. \tag{8}$$

Let (x, z, θ) the solution of (4)–(5). First, we study the qualitative properties of the curve $\alpha(s)$, specially about its curvature θ', and next we shall summarize the obtained results.

The function $z = z(s)$ satisfies $z(0) = z_0$ and $z'(0) = 0$. Since $z_0 < a/2 < -2b/a$, $\theta'(0) < 0$. Then $z''(0) < 0$. It follows that z' and z'' are negative for $s > 0$ near 0.

Claim 1: The function z satisfies $z'(s) < 0$, for any $s > 0$.

By contradiction, we assume that $s_1 > s_0$ is the first point where $z'(s_1) = 0$. In particular, $z''(s_1) \geq 0$. The function θ' is negative in the interval $[0, s_1]$. The proof is as follows. At $s = 0$, $\theta'(0) < 0$. Suppose that $\bar{s}$ is the first zero of θ', $0 < \bar{s} \leq s_1$. Define the function $f(z_0) := z_0^2 - az_0 - b$. Using the fact that $\Delta < 0$, one can show that f is always positive with a unique minimum at $z_0 = a/2$. Then (6) implies that

$$\frac{a^2 + 4b}{a^2} z(\bar{s})^2 = -f(z_0) < f(a/2) = \frac{a^2 + 4b}{4}.$$

It follows that $z(\bar{s})^2 > a^2/4 > z_0^2$, which it is a contradiction because $z(s)$ is decreasing in the interval $[0, s_1]$. As a conclusion, $\theta'(s) < 0$ in $[0, s_1]$.

We return with the function z''. As $z'(s_1) = 0$, then $\cos\theta(s_1) = \pm 1$, but $z''(s_1) = \theta'(s_1)\cos\theta(s_1) \geq 0$ implies that $\cos\theta(s_1) = -1$. By using (7),

$$z(s_1)^2 + az(s_1) + az_0 - z_0^2 = 0,$$

which yields $z(s_1) = -z_0$ or $z(s_1) = -a + z_0 < 0$, contradiction. This shows the Claim.

Once the Claim is proved, we show that $\theta' \neq 0$ for all s. If the numerator of θ' is zero for the first zero $s = s_2$, then $\cos\theta(s_2) = 2z(s_2)/a$. We use (6) to deduce that

$$\frac{a^2 + 4b}{a^2} z(s_2)^2 = -f(z_0) < -f(a/2) = \frac{a^2 + 4b}{4}.$$

Then $z(s_2)^2 > a^2/4$. But as $z(s)$ is strictly decreasing, $z(s_2)^2 < z_0^2$: this is a contradiction to the fact that $z_0 < a/2$. Therefore $\theta' < 0$, that is, θ is a decreasing and negative function for $s > 0$.

We show that $z'' < 0$ in all the domain. Assume that for some $s > 0$, $z''(s) = 0$. Since $\theta' < 0$, it follows that $\cos\theta(s) = 0$. As $a\cos\theta(s) - 2z(s) > 0$ for all s, we obtain a contradiction again.

We study the maximal domain of definition of the given solution (x, z, θ). From the above reasoning, we have two possibilities: either $z = 0$ at some point, that is, the curve α meets the x-axis, or the maximal interval $[0, s_1)$ satisfies $s_1 < \infty$ and $\lim_{s\to s_1}\theta'(s) := \theta_1 = -\infty$ with $\lim_{s\to s_1} z(s_1) := z_1 > 0$.

We see that the last one is impossible. In such case, $az_1 + 2b\cos\theta_1 = 0$ and using (6) we have,

$$(a^2+4b)z_1^2 - 4bf(z_0) = 0 \quad \text{or} \quad z_1^2 = \frac{4bf(z_0)}{a^2+4b} > \frac{4bf(a/2)}{a^2+4b} = -b.$$

Since $z(s)$ is a decreasing function on s, we deduce that

$$-b < z_1^2 < z_0^2 < a^2/4,$$

which yields a contradiction. As a conclusion, the function $z(s)$ vanishes at a first point s_1: $z(s_1) = 0$. We know that the curve α cannot be defined beyond $s = s_1$. Moreover, $x'(s) \neq 0$ for any s.

THEOREM 3.1: *Let $\alpha = \alpha(s) = (x(s), 0, z(s))$ be the profile curve of a rotational hyperbolic LW-surface S where α is the solution of* (4)–(5). *Assume that the initial condition on z_0 satisfies (8). Then*

1. *The curve α is a graph on some bounded interval $(-x_1, x_1)$ of the x-axis. In particular, α is embedded.*
2. *The curve α intersects the axis of rotation at $x = \pm x_1$.*
3. *The curve α is concave, with exactly one maximum.*

THEOREM 3.2: *Let S be a rotational hyperbolic LW-surfaces whose profile curve α satisfies the hypothesis of Theorem 3.1. Then S has the following properties:*

1. *The surface is embedded.*
2. *The Gaussian curvature of S is positive.*
3. *The surface S can not to extend to be complete. Moreover, S has exactly two singular points which coincide with the intersection of S with its axis of rotation.*

Proof. We only point out that K is positive; because in the expression (2) for the Gauss curvature K, $\cos\theta > 0$ and $\theta' < 0$. ∎

In Figure 3 (a), we show a picture of a profile curve α. The surfaces that generate behave like the surfaces of revolution with positive constant Gauss curvature. See [5], [6].

4. The case $z_0 = a/2$: cylinders

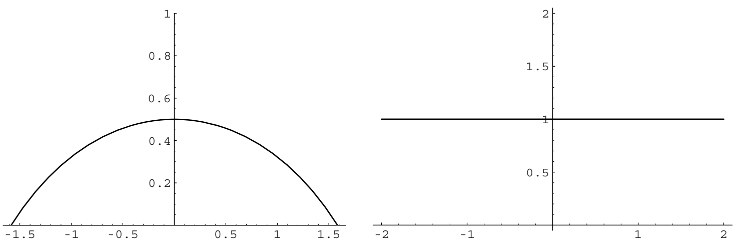
THEOREM 4.1: *Let $\alpha = \alpha(s) = (x(s), 0, z(s))$ be the profile curve of a rotational hyperbolic LW-surface S where α is the solution of* (4)–(5). *Assume that the initial condition on z_0 satisfies*

$$z_0 = a/2. \tag{9}$$

Then α is a horizontal straight-line and S is a right cylinder.

Proof. It is immediate that $x(s) = s$, $z(s) = a/2$ and $\theta(s) = 0$ is the solution of (4)–(5). ∎

See Figure 3 (b).



(a) (b)

Figure 3. Two profile curves corresponding to rotational hyperbolic LW-surfaces. We assume that $a = -b = 2$ in (1). (a) Case $z_0 = 0.5$. The maximal domain of the solution is approximately $(-1.69, 1.69)$. The curve is concave with one maximum; (b) Case $z_0 = 1$. The solution is a horizontal straight-line and the surface that generates is a right-cylinder.

5. The case $a/2 < z_0 < -2b/a$

We study the properties of the solutions of (4)–(5) when the initial condition z_0 satisfies

$$a/2 < z_0 < -2b/a. \tag{10}$$

Under this hypothesis, the value of θ' at $s = 0$ in the expression (4) is positive. Since $\theta'(0) > 0$, the function z is strictly increasing at $s = 0$. We prove that

$z'(s) > 0$ for any $s > 0$. On the contrary, if s_1 is the first point where $z'(s_1) = 0$, we have $z''(s_1) \leq 0$ and $z(s)$ is strictly increasing in $[0, s_1)$. The numerator of θ' in (4) does not vanish in the interval $[0, s_1]$ because in such case, if $a \cos\theta(\bar{s}) - 2z(\bar{s}) = 0$ for some $\bar{s}$, $0 < \bar{s} \leq s_1$, then $z(\bar{s}) \leq a/2$, a contradiction since $z_0 < z(\bar{s})$. On the other hand, in the interval $[0, s_1]$, the function $\cos\theta(s)$ does not vanish in $[0, s_1]$: if $\cos\theta(s) = 0$ for some s, then $\theta'(s) = -a/2 < 0$. As a conclusion, $z''(s_1) = \theta'(s_1)\cos\theta(s_1) > 0$, a contradiction.

With the same reasoning, one shows that the functions z' and z'' are positive in its maximal domain $[0, s_1)$. We prove that s_1 must be finite. The proof is by contradiction. Assume $s_1 = \infty$. Then

$$\lim_{s\to\infty} z(s) = \infty. \tag{11}$$

For $s = 0$, the value of the denominator of θ' in (4) is $az_0 + 2b$, which it is negative. However, using (11), $az(s) + 2b\cos\theta(s) \to \infty$ as $s \to \infty$. This means that the denominator of θ' in (4) must vanish at some point, a contradiction.

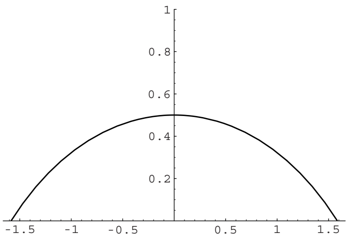
Theorem 5.1: *Let $\alpha = \alpha(s) = (x(s), 0, z(s))$ be the profile curve of a rotational hyperbolic LW-surface S where α is the solution of* (4)–(5). *Assume that the initial condition on z_0 satisfies* (10). *Then*

1. *The curve α is a graph on some bounded interval $(-x_1, x_1)$ of the x-axis. In particular, α is embedded.*
2. *The curve α is convex, with exactly one minimum.*

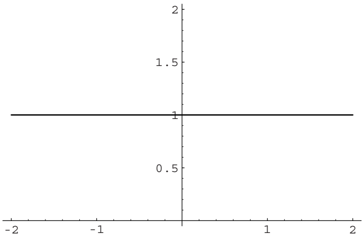
In Figure 4 (a), we present the profile curve α of a surface corresponding to the case studied in this section. As both $\cos\theta(s)$ and $\theta'(s)$ are positive functions, the expression of the Gaussian curvature K in (2) is negative.

Theorem 5.2: *Let S be a rotational hyperbolic LW-surfaces whose profile curve α satisfies the hypothesis of Theorem 5.1. Then S has the following properties:*

1. *The surface S is embedded.*
2. *The Gaussian curvature of S is negative.*
3. *The surface S cannot be extended to be complete.*



(a)



(b)

Figure 4. Two profile curves corresponding to rotational hyperbolic LW-surfaces. We assume that $a = -b = 2$ in (1). (a) Case $z_0 = 1.5$. The domain of the solution is approximately $(-0.372, 0.372)$. Here α is convex with one minimum; (b) Case $z_0 = 3$. The curve α is periodic with self-intersections.

6. The case $z_0 > -2b/a$: complete and periodic surfaces

In this section, we study the initial value problem (4)–(5) under the assumption

$$z_0 > (-2b)/a. \tag{12}$$

From (6), we write the function $z = z(s)$ as

$$z(s) = \frac{1}{2}\left(a\cos\theta(s) + \sqrt{(a^2+4b)\cos^2\theta(s) + 4(z_0^2 - az_0 - b)}\right). \tag{13}$$

Lemma 6.1: *The maximal interval of the solution* (x, z, θ) *of* (4)–(5) *is* $\mathbb{R}$.

Proof. The result follows if we prove that the derivatives x', z' and θ' are bounded. In view of (4), it suffices to show it for θ': we shall find negative numbers m and M such that $m \leq \theta'(s) \leq M < 0$ for all s. We remark that $\theta'(0) < 0$.

First, we show the existence of constants δ_1 and η_1 independent on s, with $\eta_1 < 0 < \delta_1$, such that

$$az(s) + 2b\cos\theta(s) \geq \delta_1 \quad \text{and} \quad a\cos\theta(s) - 2z(s) \geq \eta_1. \tag{14}$$

Once this is proved, it follows from (4) that

$$\theta'(s) \geq \eta_1/\delta_1 := m. \tag{15}$$

Since the function $f = f(z_0)$ is strictly increasing on z_0 for $z_0 > a/2$, there exists $\epsilon > 0$ such that

$$z_0^2 - az_0 - b = f(-2b/a) + \epsilon = \frac{b(a^2+4b)}{a^2} + \epsilon.$$

From (13),

$$\begin{aligned} z &\geq \frac{1}{2}\left(a\cos\theta + \sqrt{(a^2+4b)\cos^2\theta + \frac{4b}{a^2}(a^2+4b) + 4\epsilon}\right) \\ &\geq \frac{1}{2}\left(a\cos\theta - \frac{a^2+4b}{a} + \epsilon'\right), \end{aligned}$$

for a certain positive number ϵ'. By using the hyperbolicity condition $\Delta < 0$, we have

$$az(s) + 2b\cos\theta(s) \geq \frac{a^2+4b}{2}(\cos\theta(s) - 1) + \frac{a}{2}\epsilon' \geq \frac{a}{2}\epsilon' := \delta_1.$$

On the other hand, and using (13) again

$$a\cos\theta(s) - 2z(s) \geq -\sqrt{(a^2+4b)\cos^2(s)\theta + 4f(z_0)} \geq -2\sqrt{f(z_0)} := \eta_1.$$

We now obtain the upper bound M for θ'. We prove that there exist δ_2, η_2, with $\eta_2 < 0 < \delta_2$ such that

$$az(s) + 2b\cos\theta(s) \leq \delta_2 \quad \text{and} \quad a\cos\theta(s) - 2z(s) \leq \eta_2. \tag{16}$$

Using (13),

$$\begin{aligned} az(s) + 2b\cos\theta(s) &= \frac{1}{2}\left((a^2+4b)\cos\theta(s) + a\sqrt{(a^2+4b)\cos^2\theta(s) + 4f(z_0)}\right) \\ &\leq a\sqrt{f(z_0)} := \delta_2. \end{aligned}$$

On the other hand,

$$\begin{aligned} a\cos\theta(s) - 2z(s) &= -\sqrt{(a^2+4b)\cos^2\theta(s) + 4f(z_0)} \\ &\leq -\sqrt{(a^2+4b) + 4f(-2b/a)} := \eta_2. \end{aligned}$$

Hence, we deduce from (4) that

$$\theta'(s) \leq \eta_2/\delta_2 := M. \tag{17}$$

The inequalities (15) and (17) concludes the proof of the lemma. ∎

As a consequence of the proof of Lemma 6.1, the graphic of the function θ lies between two tilted straight-lines. Since the derivative of θ is negative, $\theta(s)$ is strictly decreasing with

$$\lim_{s\to\infty} \theta(s) = -\infty.$$

Put $T > 0$ the first number such that $\theta(T) = -2\pi$. We prove that α is a periodic curve.

LEMMA 6.2: *Under the hypothesis of this section and with the above notation, we have:*

$$x(s+T) = x(s) + x(T)$$
$$z(s+T) = z(s)$$
$$\theta(s+T) = \theta(s) - 2\pi$$

Proof. This is a consequence of the uniqueness of solutions of (4)–(5). We only have to show that $z(T) = z_0$. But this direct from (13), the assumption (12) and that $a/2 < -2b/a$. ∎

As conclusion of Lemmas 6.1 and 6.2, we describe the behavior of the coordinates functions of the profile curve α under the assumption (12). See Figure 4 (b). Due to the monotonicity of θ, let T_1, T_2 and T_3 be the points in the period $[0,T]$ such that the function θ takes the values $-\pi/2, -\pi$ and $-3\pi/2$, respectively. In view of the variation of the angle θ with the time coordinate s, it is easy to verify the following Table:

s	θ	$x(s)$	$z(s)$
$[0,T_1]$	$[0,\frac{-\pi}{2}]$	increasing	decreasing
$[T_1,T_2]$	$[\frac{-\pi}{2},-\pi]$	decreasing	decreasing
$[T_2,T_3]$	$[-\pi,\frac{-3\pi}{2}]$	decreasing	increasing
$[T_3,T]$	$[\frac{-3\pi}{2},-2\pi]$	increasing	increasing

THEOREM 6.3: *Let $\alpha = \alpha(s) = (x(s), 0, z(s))$ be the profile curve of a rotational hyperbolic LW-surface S where α is the solution of* (4)–(5). *Assume that the initial condition on z_0 satisfies* (12). *Then*

1. *The curve α is invariant by the group of translations in the x-direction given by the vector $(x(T), 0, 0)$.*

2. *In each period of z, the curve α presents one maximum at $s = 0$ and one minimum at $s = T_2$. Moreover, α is symmetric with respect to the vertical line at $x = 0$ and $x = x(T_2)$.*
3. *The height function of α, that is, $z = z(s)$, is periodic.*
4. *The curve α has self-intersections and its curvature has constant sign.*
5. *The part of α between the maximum and the minimum satisfies that the function $z(s)$ is strictly decreasing with exactly one vertical point. Between this minimum and the next maximum, $z = z(s)$ is strictly increasing with exactly one vertical point.*
6. *The velocity α' turns around the origin.*

THEOREM 6.4: *Let S be a rotational hyperbolic LW-surfaces whose profile curve α satisfies the hypothesis of Theorem 6.3. Then S has the following properties:*

1. *The surface has self-intersections.*
2. *The surface is periodic with infinite vertical symmetries.*
3. *The surface is complete.*
4. *The part of α between two consecutive vertical points and containing a maximum corresponds with points of S with positive Gaussian curvature; on the other hand, if this part contains a minimum, the Gaussian curvature is negative in this set of the surface.*

COROLLARY 6.5: *Let $\alpha(s) = (x(s), 0, z(s))$ be the profile curve of a rotational hyperbolic LW-surface. Assume that α is the solution of (4)–(5) where $z_0 > -2b/a$. Then the graphic of α lies between the lines $z = z_0 - a$ and $z = z_0$, that is,*

$$z_0 - a \leq z(s) \leq z_0,$$

where $z(s)$ reaches the minimum and the maximum values in a discrete set of points.

Proof. Since $\theta \to -\infty$, the minimum and the maximum of the function $z(s)$ reach at those points with $\cos\theta = -1$ and $\cos\theta = 1$, respectively. The estimate is obtained from (13). ∎

As it was announced in the Introduction and with the purpose to distinguish with the surfaces of negative constant Gaussian curvature, we stand out from Theorem 6.4 the following

COROLLARY 6.6: *There exists a one-parameter family of rotational hyperbolic linear Weingarten surfaces that are complete and with self-intersections in* $\mathbb{R}^3$. *Moreover, the generating curves of these surfaces are periodic.*

7. A new family of LW-surfaces of hyperbolic type

In the previous sections, we have considered the initial condition $\theta(0) = 0$ in (5) on the starting angle of the profile curve α. This corresponds with the fact that the tangent line to α at the initial point is parallel to the axis of revolution of the surface that generates. However, this condition can be substituted for another one, namely, $\theta(0) = \theta_0$. This means that the new profile curve α could not have points in all its maximal domain I whose tangent line is horizontal. Following the notation of the paper, this is written as $z'(s) \neq 0$ for any $s \in I$. This section is devoted to obtain examples of such surfaces.

We take $\theta_0 = -\pi/2$ as starting angle and so, we consider solutions

$$\{x(s), z(s), \theta(s)\}$$

of the differential equations system (4) subject the conditions

$$x(0) = 0, \quad z(0) = z_0, \quad \theta(0) = -\pi/2. \tag{18}$$

We have then (see Figure 5):

THEOREM 7.1: *Let* $\alpha = \alpha(s) = (x(s), 0, z(s))$ *be the profile curve of a rotational hyperbolic LW-surface* S *where* α *is the solution of* (4)–*18.*

1. *If* $z_0 < \sqrt{-b}$, *the curve* α *intersects transversally the* x*-axis. The maximal interval of* α *is bounded.*
2. *If* $z_0 = \sqrt{-b}$, *the curve* α *intersect tangentially the* x*-axis.*
3. *If* $z_0 > \sqrt{-b}$, *the curve* α *does not intersect the* x*-axis. In this case, the curve* α *is one of the solutions obtained in Theorem 6.4.*

Proof. The integral (6) is now

$$z(s)^2 - az(s)\cos\theta(s) - b\cos^2\theta(s) - z_0^2 = 0. \tag{19}$$

Thus, if $z(s) = 0$ at some point s, we have $z_0^2 = -b\cos^2\theta(s)$. This implies $z_0^2 \leq -b$. As a consequence, if $z_0 > \sqrt{-b}$, the curve α does not intersect the x-axis.

At $s = 0$, we know that $\theta'(0) = -2/a$ and so, θ is strictly decreasing for a certain interval $[0, \delta)$ of $s = 0$. In particular, $-1 \leq \sin\theta(s) < 0$ for $0 \leq s < \delta$

and z is a decreasing function. We see that the numerator and denominator of θ' are negative and positive respectively for numbers $s > 0$ close to $s = 0$. On the other hand, θ' vanishes at the points that satisfy $\cos\theta(s) = 2z(s)/a$ and the denominator in those points with $\cos\theta(s) = -az(s)/2b$. As $\Delta < 0$, we deduce that $(-az(s))/2b < (2z(s))/a$. We prove that whenever $z(s) > 0$ and $\sin\theta(s) < 0$, the function $az(s) + 2b\cos\theta(s)$ is increasing. This is a direct consequence of the computation of its derivative, namely,

$$\frac{(a^2+4b)\sin\theta(s)}{az(s)+2b\cos\theta(s)}z(s) > 0.$$

Thus we bound as follows: $az(s) + 2b\cos\theta(s) \geq az_0$. Since z is decreasing, then $a\cos\theta(s) - 2z(s) \geq -a - 2z_0$. Therefore,

$$0 > \theta'(s) \geq -(a+2z_0)/az_0 > -\infty.$$

This means that we can continue the solution $\alpha(s)$ provided $z(s) > 0$ and $\sin\theta(s) < 0$. Moreover, and since z is decreasing near $s = 0$ and using (19), $z(s)$ and $\theta(s)$ are decreasing functions, at least until that θ reaches the value $-\pi$. We consider the three cases described in the statement of Theorem 7.1.

1. Case $z_0 < \sqrt{-b}$. We have two possibilities. First, there exists $s_0 > 0$ such that $z'(s_0) = 0$ (and so, $\theta(s_0) = -\pi$). From (19) and using $z_0^2 < -b$, we have $z(s_0) < 0$: contradiction. Therefore, the only possibility is that $z'(s) \neq 0$ for any s and $z(s)$ is strictly decreasing. In particular, α is defined for any s and

$$\lim_{s\to\infty} z(s) = z_1 \geq 0, \quad \lim_{s\to\infty} \theta'(s) = 0.$$

   But letting $s \to \infty$ in the expression of θ' in (4), we obtain

$$\theta'(s) \to -\frac{a+2z_1}{az_1-2b} \neq 0.$$

   This contradiction implies that z must meet the x-axis. Furthermore, this intersection must be transversal by using (19) again.
2. Case $z_0^2 = -b$. From (19), the function z takes the value 0 if and only if $\cos\theta(s) = \pm 1$. As $z(s)$ is decreasing in a certain interval on the right of $s = 0$, θ is decreasing. Thus, if $z(s) > 0$, we have then that either $z'(s_0) = 0$ at some point s_0, which implies by using (19) that $z(s_0) = 0$ and $\theta(s_0) = -\pi$ or the solution α is defined for any $s > 0$ and $z(s)$ is a decreasing function. In such case, both z and θ are asymptotic to horizontal lines and

$$\lim_{s\to\infty} z(s) = z_1 \geq 0, \quad \lim_{s\to\infty} \theta'(s) = 0.$$

Now (19) gives $z_1 = 0$. Returning to the expression of θ' in (4), we have that $\theta'(s) \to a/2b \neq 0$ as $s \to \infty$. This yields a contradiction and the second option is impossible.

3. Case $z_0^2 > -b$. We show that z' vanishes at some point. On the contrary, the solution α is defined for any s with $\theta'(s) \to 0$ as $s \to \infty$. If $\epsilon > 0$ is the number such that $z_0^2 = -b + \epsilon$, we have from (19) that

$$z(s) = \frac{1}{2}\left(a\cos\theta(s) + \sqrt{(a^2+4b)\cos^2\theta(s) + 4z_0^2}\right)$$
$$\to \frac{1}{2}\left(-a + \sqrt{a^2+\epsilon}\right) := z_1 > 0.$$

But (4) implies then that $\theta'(s) \to -(a+2z_1)/(az_1 - b) \neq 0$ again. This contradiction means that there exists $s_1 > 0$ such that $z'(s_1) = 0$, and so, $\theta(s_1) = -\pi$. As a consequence, the solution α presents a horizontal tangent line at the point $s = s_1$ whose velocity vector is $\alpha'(s_1) = (-1, 0, 0)$, such as it happens with the solutions given in Theorem 6.4: this point corresponds with the minimum of the z-function. The uniqueness of solutions of a system of differential equations implies that our solution must be one of the obtained there. ∎

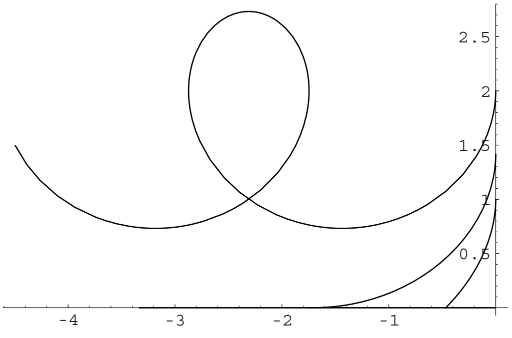


Figure 5. Three profile curves corresponding to rotational hyperbolic LW-surfaces whose starting angle is $\theta(0) = -\pi/2$. We assume that $a = -b = 2$ in (1). Each one of the curves corresponds with the initial condition $z_0 = 1$, $z_0 = \sqrt{2}$ and $z_0 = 2$. They comprise all cases described in Theorem 7.1.

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